Four-dimensional double singular oscillator

Mara Petrosyan

Artsakh State University, Stepanakert and Yerevan State University, Yerevan, Armenia

Abstract

The Schrödinger equation for the four-dimensional double singular oscillator is separable in Eulerian, doble polar and spheroidal coordinates in $\mathbb{R}^4$. It is shown that the coefficients for the expansion of double polar basis in terms of the Eulerian basis can be expressed through the Clebsch-Gordan coefficients of the group $SU(2)$ analytically continued to real values of their arguments. The coefficients for the expansions of the spheroidal basis in terms of the Eulerian and double polar bases are proved to satisfy three-term recursion relations.

1 Introduction

The Schrödinger equation for a four-dimensional double singular oscillator has the form [1]

$$\hat{H}\psi = -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial u_i^2} + \left(\frac{\mu\omega^2 u^2}{2} + \frac{c_1}{u_0^2+u_1^2} + \frac{c_2}{u_2^2+u_3^2}\right)\psi = E\psi, \tag{1}$$

where u_i $(i = 0, 1, 2, 3)$ are Cartesian coordinates of the space $\mathbb{R}^4$ and c_1 and c_2 nonnegative constants.

In our paper [1] it is shown that the four-dimensional double singular oscillator and generakized MIC-Kepler problem decribed by the Schrödinger equation [2]

$$\frac{1}{2\mu}\left(-i\nabla\frac{e}{c}s\mathbf{A}\right)^2\psi + \left[\frac{\hbar^2 s^2}{2\mu r^2} - \frac{2}{r} + \frac{\lambda_1}{r(r+z)} + \frac{\lambda_2}{r(r-z)}\right]\psi = \epsilon\psi, \tag{2}$$

are dual. Here the vector potential $\mathbf{A}$ corresponds to the Dirac monopole with the magnetic charge $g = \hbar cs/e$ $(s = 0, \pm 1/2, \pm 1, \ldots)$ and has the form

$$\mathbf{A} = \frac{1}{r(r-z)}(y, -x, 0), \qquad \text{and} \qquad \mathrm{rot}\mathbf{A} = \frac{\mathbf{r}}{r^3}.$$

The transformation of duality is the generalization version of the so-called Kustaanheimo-Stiefel transformation

$$\begin{aligned} x + iy &= 2\left(u_0 + iu_1\right)\left(u_2 + iu_3\right), \\ z &= u_0^2 + u_1^2 - u_2^2 - u_3^2 \\ \gamma &= \frac{i}{2}\ln\frac{\left(u_0 - iu_1\right)\left(u_2 + iu_3\right)}{\left(u_0 + iu_1\right)\left(u_2 - iu_3\right)} \end{aligned} \tag{3}$$

supplemented with the ansatz $s \to -i\partial/\partial\gamma$. We also noted, that the parameters of these systems are connected with each other by the relations:

$$E = 4e^2, \qquad \epsilon = -\frac{\mu\omega^2}{8}, \qquad c_a = 2\lambda_a, \qquad \text{where} \qquad a = 1, 2.$$

The first two lines of (3) are the transformation $\mathbb{R}^4 \to \mathbb{R}^3$ suggested by Kustaanheimo and Stiefel for regularization of the equations of the celestial mechanics [3]. Later, this transformation found other applications as well [4, 5]. The generalized Kustaanheimo-Stiefel transformation (3) (the Kustaanheimo-Stiefel transformation supplemented with the angle γ) was used for "synthesis" of the charge-dyon system from the four-dimensional isotropic oscillator [6].

It should be noted that equation (2) for $\lambda_a = 0$ and $s \neq 0$ reduces tj the Schrödinger equation for the MIC-Kepler system [7, 8]. At $s = 0$, the Schrödinger equation (2) is reduced to the Schrödinger equation for the generalized Kepler-Coulomb system [9]. In the case when $s = 0$ and $\lambda_1 = \lambda_2 \neq 0$, Eq. (2) reduces to the system suggested by Hartmann, at was used for explanation of the spectrum of the benzene molecule [10, 11, 12].

2 Eulerian and Double Polar Bases

Determine the Eulerian coordinates in $\mathbb{R}^4$ as follows:

$$u_0 + iu_1 = u\,\cos\frac{\beta}{2}\,e^{i\frac{\alpha+\gamma}{2}}, \qquad u_2 + iu_3 = u\,\sin\frac{\beta}{2}\,e^{i\frac{\alpha-\gamma}{2}}, \tag{4}$$

where $u \in [0,\infty)$, $\alpha \in [0,2\pi)$, $\beta \in [0,\pi]$, $\gamma \in [0,4\pi)$. In these coordinates the differential elements of length and volume, and the Laplace operator have the form

$$\begin{aligned}
dl^2 &= du^2 + \frac{u^2}{4}\left(d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta\,d\alpha\,d\gamma\right),\\
dV &= \frac{u^3}{8}\,\sin\beta\,du\,d\alpha\,d\beta\,d\gamma,\\
\frac{\partial^2}{\partial u_i^2} &= \frac{1}{u^3}\frac{\partial}{\partial u}\left(u^3\,\frac{\partial}{\partial u}\right) - \frac{4}{u^2}\,\hat{J}^2,
\end{aligned}$$

where

$$\hat{J}^2 = -\left[\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\left(\sin\beta\,\frac{\partial}{\partial\beta}\right) + \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\cos\beta\,\frac{\partial^2}{\partial\alpha\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right)\right]$$

is the square of the angular momentum operator.

The Schrödinger equation (1) in the Eulerian coordinates (4) may be solved by seeking a wave function ψ of the form

$$\psi\left(u,\alpha,\beta,\gamma\right) = R\left(u\right)\,Z\left(\alpha,\beta,\gamma\right). \tag{5}$$

This amouns to finding the eigenfunctions of $\left\{\hat{H},\,\hat{\Lambda},\,\hat{J}_3,\,\hat{J}_3'\right\}$ of commuting operators, where the constant of motion $\hat{\Lambda}$ reads

$$\hat{\Lambda} = \hat{J}^2 + \frac{\mu}{\hbar^2}\left(\frac{c_1}{1+\cos\beta} + \frac{c_2}{1-\cos\beta}\right) \tag{6}$$

and which in the Cartesian coordinates u_i has the form

$$\hat{\Lambda} = -\frac{1}{4}\left(u^2\frac{\partial^2}{\partial u_i^2} - u_iu_j\frac{\partial^2}{\partial u_i\partial u_j} - 3u_i\frac{\partial}{\partial u_i}\right) + \frac{\mu u^2}{2\hbar^2}\left(\frac{c_1}{u_0^2+u_1^2} + \frac{c_2}{u_2^2+u_3^2}\right). \tag{7}$$

The operators $\hat{J}_3$ and $\hat{J}_3'$ are defined as follows:

$$\hat{J}_3\,\psi = -\frac{\partial\psi}{\partial\alpha} = m\psi, \qquad \hat{J}_3'\,\psi = -\frac{\partial\psi}{\partial\gamma} = s\psi.$$

After substitution the expression (5) into the Eq. (1) the variables in the Schrödinger equation (1) are separated and we arrive at the following system of coupled differential equations:

$$\begin{aligned}
&\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\left(\sin\beta\,\frac{\partial Z}{\partial\beta}\right) + \frac{1}{2\left(1+\cos\beta\,\right)}\left[\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\gamma}\right)^2 - \frac{2\mu c_1}{\hbar^2}\right]Z +\\
&+\frac{1}{2\left(1-\cos\beta\,\right)}\left[\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\gamma}\right)^2 - \frac{2\mu c_2}{\hbar^2}\right]Z = -\lambda Z,
\end{aligned} \tag{8}$$

$$\frac{1}{u^3}\frac{d}{du}\left(u^3\,\frac{dR}{du}\right) - \frac{4\lambda}{u^2}\,R + \frac{2\mu}{\hbar^2}\left(E - \frac{\mu\omega^2u^2}{2}\right)R = 0, \tag{9}$$

where λ is the separation constant which is the eigenvalue of the operator (6).

The solution of Eq. (8) is easily found to be

$$Z_{jms}\left(\alpha,\beta,\gamma;\delta_1,\delta_2\right)=N_{jms}\left(\delta_1,\delta_2\right)\left(\cos\frac{\beta}{2}\right)^{m_1}\left(\sin\frac{\beta}{2}\right)^{m_2}P_{j-m_+}^{(m_2,m_1)}(\cos\beta)e^{im\alpha}\,e^{is\gamma}, \tag{10}$$

where $m_{1,2}=|m\pm s|+\delta_{1,2}=\sqrt{(m\pm s)^2+2\mu c_{1,2}/\hbar^2}$, $m_+=(|m+s|+|m-s|)/2$ and $P_n^{(a,b)}(x)$ denotes a Jacobi polynomial. The quantum number j characterizes the total angular momentum and for the (half)integer j the quantum numbers m and s are (half)integer. At a fixed value j the m and s run through values: $m,s=-j,-j+1,\ldots,j-1,j$.

Farthemore, the separation constant λ is quantized as

$$\lambda=\left(j+\frac{\delta_1+\delta_2}{2}\right)\left(j+\frac{\delta_1+\delta_2}{2}+1\right). \tag{11}$$

The normalization constant $N_{jm}(\delta_1,\delta_2)$ in (10) is given (up to a phase factor) by

$$N_{jms}(\delta_1,\delta_2)=(-1)^{\frac{m-s+|m-s|}{2}}\sqrt{\frac{(2j+\delta_1+\delta_2+2)(j-m_+)!\Gamma(j+m_++\delta_1+\delta_2+1)}{16\pi^2\,\Gamma(j+m_-+\delta_1+1)\Gamma(j-m_-+\delta_2+1)}},$$

where $m_-=(|m+s|-|m-s|)/2$ and we assume that

$$\frac{1}{8}\int\limits_0^{\pi}\int\limits_0^{2\pi}\int\limits_0^{4\pi}\sin\beta\,Z^*_{j'm's'}\left(\alpha,\beta,\gamma;\delta_1,\delta_2\right)Z_{jms}\left(\alpha,\beta,\gamma;\delta_1,\delta_2\right)d\alpha\,d\beta\,d\gamma=\delta_{jj'}\delta_{mm'}\delta_{ss'}.$$

Let us go now to the radial equation (9). The introduction of (11) into (9) leads to

$$\frac{1}{u^3}\frac{d}{du}\left(u^3\frac{dR}{du}\right)-\frac{1}{u^2}\left(2j+\delta_1+\delta_2\right)\left(2j+\delta_1+\delta_2+2\right)R+\frac{2\mu}{\hbar^2}\left(E-\frac{\mu\omega^2u^2}{2}\right)R=0.$$

The solution of this equation normalized by the condition

$$\int\limits_0^{\infty}u^3\,R_{Nj}\left(u;\delta_1,\delta_2\right)R_{N'j}\left(u;\delta_1,\delta_2\right)du=\delta_{NN'} \tag{12}$$

has the form

$$R_{Nj}(u;\delta_1,\delta_2)=C_{Nj}(\delta_1,\delta_2)\,(au)^{2j+\delta_1+\delta_2}\,e^{-\frac{a^2u^2}{2}}F\left(-\frac{N}{2}+j;2j+\delta_1+\delta_2+2;a^2u^2\right), \tag{13}$$

where $F(a;c;x)$ is the confluent hypergeometric function, $a=\sqrt{\mu\omega/\hbar}$ and

$$C_{Nj}(\delta_1,\delta_2)=\frac{4a^2}{\Gamma\left(2j+\delta_1+\delta_2+2\right)}\sqrt{\frac{\Gamma\left(\frac{N}{2}+j+\delta_1+\delta_2+2\right)}{(\frac{N}{2}-j)!}}$$

The energy spectrum has the form

$$E_N=\hbar\omega\,\left(N+\delta_1+\delta_2+2\right) \tag{14}$$

where N is the principle quantum number, $N=0,1,2,\ldots$, and the quantum number j run through the values: $j=m_+,m_++1,\ldots,N/2$.

Thus, the Eulerian basis (5) is the eigenfunction of the operator (6) and

$$\hat{\Lambda}\,\psi_{Njms}=\left(j+\frac{\delta_1+\delta_2}{2}\right)\left(j+\frac{\delta_1+\delta_2}{2}+1\right)\psi_{Njms}. \tag{15}$$

In the limiting case $\delta_1 = \delta_2 = 0$ we recover the familiar results for the four-dimensional isotropic oscillator [14].

Let us consider the four-dimensional double singular oscillator in the double polar coordinates [14]. In the double polar coordinates $\rho_1, \rho_2 \in [0, \infty)$, $\varphi_1, \varphi_2 \in [0, 2\pi)$, defined by the formulae

$$u_0 + iu_1 = \rho_1 \, e^{i\varphi_1}, \qquad u_2 + iu_3 = \rho_2 \, e^{i\varphi_2}, \tag{16}$$

the differential elements of length and volume read

$$dl^2 = d\rho_1^2 + d\rho_2^2 + \rho_1^2 \, d\varphi_1^2 + \rho_2^2 \, d\varphi_2^2, \qquad dV = \rho_1 \, \rho_2 \, d\varphi_1 \, d\varphi_2,$$

while the Laplace operator looks like

$$\frac{\partial^2}{\partial u_i^2} = \frac{1}{\rho_1} \frac{\partial}{\partial \rho_1} \left(\rho_1 \frac{\partial}{\partial \rho_1} \right) + \frac{1}{\rho_2} \frac{\partial}{\partial \rho_2} \left(\rho_2 \frac{\partial}{\partial \rho_2} \right) + \frac{1}{\rho_1^2} \frac{\partial^2}{\partial \varphi_1^2} + \frac{1}{\rho_2^2} \frac{\partial^2}{\partial \varphi_2^2}.$$

The substitution

$$\psi(\rho_1, \rho_2, \varphi_1, \varphi_2) = \frac{1}{2\pi} \Phi_1(\rho_1) \Phi_2(\rho_2) \, e^{iM_1\varphi_1} \, e^{iM_2\varphi_2}$$

where $M_1, M_2 = 0, \pm 1, \pm 2, \ldots$, separates the variables in the Schrödinger equation (1) and we arrive at the following system of equations:

$$\frac{1}{\rho_1} \frac{d}{d\rho_1} \left(\rho_1 \frac{d\Phi_1}{d\rho_1} \right) - \frac{(|M_1| + \Delta_1)^2}{\rho_1^2} + \frac{2\mu}{\hbar^2} \left(\frac{E}{2} + \frac{\hbar^2}{4\mu} \Omega - \frac{\mu\omega\rho_1^2}{2} \right) \Phi_1 \;\; = \;\; 0, \tag{17}$$

$$\frac{1}{\rho_2} \frac{d}{d\rho_2} \left(\rho_2 \frac{d\Phi_2}{d\rho_2} \right) - \frac{(|M_2| + \Delta_2)^2}{\rho_2^2} + \frac{2\mu}{\hbar^2} \left(\frac{E}{2} - \frac{\hbar^2}{4\mu} \Omega - \frac{\mu\omega\rho_2^2}{2} \right) \Phi_2 \;\; = \;\; 0, \tag{18}$$

where $\beta -$ is the separation constant and $\Delta_a = \sqrt{M_a^2 + 2\mu c_a/\hbar^2} - |M_a|$.

These equations are analogous with the equations of the circular oscillator in the polar coordinates [15]. Thus, we get

$$\psi_{N_1 N_2 M_1 M_2}(\rho_1, \rho_2, \varphi_1, \varphi_2; \delta_1, \delta_2) = \frac{1}{2\pi} \Phi_{N_1 M_1}(a^2\rho_1^2; \Delta_1) \, \Phi_{N_2 M_2}(a^2\rho_2^2; \Delta_2) \, e^{iM_1\varphi_1} \, e^{iM_2\varphi_2}, \tag{19}$$

where

$$\Phi_{N_a M_a}(x) = \sqrt{\frac{2\Gamma(N_a + |M_a| + \Delta_a + 1)}{(N_a)!}} \; \frac{a e^{-\frac{x}{2}} \; x^{(|M_a| + \Delta_a)/2}}{\Gamma\left(|M_a| + \Delta_a + 1\right)} \; F(-N_a; |M_a| + \Delta_a + 1; x).$$

Here N_1 and N_2 are nonnegative integers

$$N_1 = -\frac{|M_1| + \Delta_1 + 1}{2} + \frac{E}{4\hbar\omega} + \frac{\Omega}{8a^2}, \qquad N_2 = -\frac{|M_2| + \Delta_2 + 1}{2} + \frac{E}{4\hbar\omega} - \frac{\Omega}{8a^2}.$$

From the last relations, taking into account (14), we get that the double polar quantum numbers N_1 and N_2 are connected with the principal quantum number N as follows:

$$N = 2N_1 + 2N_2 + |M_1| + |M_2| \, .$$

Excluding the energy E from Eqs. (17) and (18), we obtain the additional integral of motion

$$\begin{aligned} \hat{\Omega} \;\; &= \;\; \left[\frac{1}{\rho_2} \frac{\partial}{\partial \rho_2} \left(\rho_2 \, \frac{\partial}{\partial \rho_2} \right) - \frac{1}{\rho_1} \frac{\partial}{\partial \rho_1} \left(\rho_1 \, \frac{\partial}{\partial \rho_1} \right) + \frac{1}{\rho_2^2} \frac{\partial^2}{\partial \varphi_2^2} - \frac{1}{\rho_1^2} \frac{\partial^2}{\partial \varphi_1^2} \right] + \\ &+ \;\; \frac{\mu^2\omega^2}{\hbar^2} \left(\rho_1^2 - \rho_2^2 \right) + \frac{2\mu}{\hbar^2} \left(\frac{c_1}{\rho_1^2} - \frac{c_2}{\rho_2^2} \right) \end{aligned} \tag{20}$$

with the eigenvalues

$$\Omega = \frac{2\mu\omega}{\hbar}\left(2N_1 - 2N_2 - |M_1| + |M_2| - \Delta_1 + \delta_2\right)$$

and eigenfunctions $\psi_{N_1 N_2 M_1 M_2}(\rho_1, \rho_2, \varphi_1, \varphi_2; \Delta_1, \Delta_2)$, i.e.

$$\hat{\Omega}\psi_{N_1 N_2 M_1 M_2}(\rho_1, \rho_2, \varphi_1, \varphi_2; \Delta_1, \Delta_2) = \Omega\psi_{N_1 N_2 M_1 M_2}(\rho_1, \rho_2, \varphi_1, \varphi_2; \Delta_1, \Delta_2). \tag{21}$$

In the Cartesian coordinates, the operator $\hat{\Omega}$ can be rewritten as

$$\begin{aligned} \hat{\Omega} &= \left(\frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} - \frac{\partial^2}{\partial u_0^2} - \frac{\partial^2}{\partial u_1^2}\right) + \\ &+ \frac{\mu^2\omega^2}{\hbar^2}\left(u_0^2 + u_1^2 - u_2^2 - u_3^2\right) + \frac{2\mu}{\hbar^2}\left(\frac{c_1}{u_0^2 + u_1^2} - \frac{c_2}{u_2^2 + u_3^2}\right). \end{aligned} \tag{22}$$

3 Connection Between Eulerian and Double Polar Bases

At fixed energy values we write down the double polar bound states (19) as a coherent quantum mixture of Eulerian bound states

$$\psi_{N_1 N_2 M_1 M_2} = \sum_{j,m,s} W^{jms}_{N_1 N_2 M_1 M_2}\left(\delta_1, \delta_2\right)\, \psi_{Njms}. \tag{23}$$

Our goal is the derivation of an explicit form of the coefficient $W^{jms}_{N_1 N_2 M_1 M_2}$. First, we should like to note that from the comparison of (4) with (16) we have

$$\rho_1 = u\cos\frac{\beta}{2}, \qquad \rho_2 = u\sin\frac{\beta}{2}, \qquad \varphi_1 = \frac{\alpha+\gamma}{2}, \qquad \varphi_2 = \frac{\alpha-\gamma}{2}. \tag{24}$$

In relation (23), according to (24), we pass from the double polar coordinates to the Eulerian ones. Then, substituting $\beta = 0$, taking account of

$$P_n^{(a,b)}(1) = \frac{(a+1)_n}{n!},$$

and using the orthogonality condition for radial wave functions in hypermomentum [14]

$$\int_0^\infty R_{Nj'}\left(u; \delta_1, \delta_2\right) R_{Nj}\left(u; \delta_1, \delta_2\right)\, u du = \frac{2a^2\, \delta_{jj'}}{(2j + \delta_1 + \delta_2 + 2)},$$

we obtain the following integral representation for the coefficients $W^{jms}_{N_1 N_2 M_1 M_2}$:

$$W^{jms}_{N_1 N_2 M_1 M_2}\left(\delta_1, \delta_2\right) = \frac{\sqrt{(2j + \delta_1 + \delta_2 + 1)\,(j - m_+)!}}{\Gamma(m_2 + 1)\Gamma(2j + \delta_1 + \delta_2 + 2)}\, E^{jms}_{N_1 N_2}\, K^{NN_1}_{jms}\, \delta_{M_1, m+s}\, \delta_{M_2, m-s}. \tag{25}$$

Here

$$\begin{aligned} E^{jms}_{N_1 N_2} &= \sqrt{\Gamma\left(\frac{N}{2} + j + \delta_1 + \delta_2 + 2\right)} \times \\ &\times \left[\frac{\Gamma\left(j - m_- + \delta_1 + 1\right)\Gamma(N_1 + m_1 + 1)\Gamma(N_2 + m_2 + 1)}{(N_1)!(N_2)!(\frac{N}{2} - j)!\Gamma\left(j + m_- + \delta_2 + 1\right)\Gamma\left(j + m_+ + \delta_1 + \delta_2 + 1\right)}\right]^{\frac{1}{2}}, \end{aligned} \tag{26}$$

and

$$K_{jms}^{Nn_1} = \int\limits_0^\infty e^{-x} x^{j+m_1+m_2+\delta_1+\delta_2} F\left(-N_1; m_1+1; x\right) F\left(-\frac{N}{2}+j; 2j+\delta_1+\delta_2+2; x\right) dx, \tag{27}$$

where $x = a^2u^2$. Further, in (27) writing down the confluent hypergeometric function $F\left(-N_1; m_1+1; x\right)$ as a polynomial, integrating by the formula [16]

$$\int\limits_0^\infty e^{-\lambda x} x^\nu F\left(\alpha; \gamma; kx\right)\, dx = \frac{\Gamma\left(\nu+1\right)}{\lambda^{\nu+1}} {}_2F_1\left(\alpha; \nu+1; \gamma\frac{k}{\lambda}\right) dx,$$

and taking account of the relation

$${}_2F_1\left(a, b; c; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

we derive

$$K_{jms}^{NN_1} = \frac{\left(\frac{N}{2}-m_+\right)!\Gamma(2j+\delta_1+\delta_2+2)\Gamma\left(j+m_++\delta_1+\delta_2+1\right)}{(j-m_+)!\Gamma\left(\frac{N}{2}+j+\delta_1+\delta_2+1\right)} \times$$
$$\times {}_3F_2\left\{\begin{array}{l} -N_1, -j+m_+, j+m_++\delta_1+\delta_2+1 \\ m_1+1, -\frac{N}{2}+m_++1 \end{array}\middle| 1\right\}. \tag{28}$$

The introduction of (26) and (28) into (25) gives

$$W_{N_1N_2M_1M_2}^{jms}\left(\delta_1, \delta_2\right) = \sqrt{\frac{\left(2j+\delta_1+\delta_2+1\right)\Gamma(N_1+m_1+1)\Gamma(N_2+m_2+1)}{(N_1)!(N_2)!\left(\frac{N}{2}-j\right)!(j-m_+)!\Gamma\left(j+m_-+\delta_2+1\right)}} \times$$
$$\times \frac{\left(\frac{N}{2}-m_+\right)!}{\Gamma(m_1+1)} \sqrt{\frac{\Gamma\left(j-m_-+\delta_1+1\right)\Gamma\left(j+m_++\delta_1+\delta_2+1\right)}{\Gamma\left(\frac{N}{2}+j+\delta_1+\delta_2+1\right)}} \times$$
$$\times {}_3F_2\left\{\begin{array}{l} -n_1, -j+m_+, j+m_++\delta_1+\delta_2+1 \\ m_1+1, -\frac{N}{2}+m_++1 \end{array}\middle| 1\right\} \delta_{M_1,m+s}\, \delta_{M_2,m-s}. \tag{29}$$

It is known that the Clebsch-Gordan coefficients for the group $SU(2)$ can be written as [9]

$$C_{a\alpha;b\beta}^{c\gamma} = \left[\frac{(2c+1)(b-a+c)!(a+\alpha)!(b+\beta)!(c+\gamma)!}{(b-\beta)!(c-\gamma)!(a+b-c)!(a-b+c)!(a+b+c+1)!}\right]^{1/2} \times$$
$$\times \delta_{\gamma,\alpha+\beta} \frac{(-1)^{a-\alpha}}{\sqrt{(a-\alpha)!}} \frac{(a+b-\gamma)!}{(b-a+\gamma)!} {}_3F_2\left\{\begin{array}{l} -a+\alpha, c+\gamma+1, -c+\gamma \\ \gamma-a-b, b-a+\gamma+1 \end{array}\middle| 1\right\}. \tag{30}$$

Finally, comparing (29) and (30), we arrive at the following representation:

$$W_{N_1N_2M_1M_2}^{jms}\left(\delta_1, \delta_2\right) = (-1)^{N_1+\frac{m-s+|m-s|}{2}}\, \delta_{M_1,m+s}\, \delta_{M_2,m-s} \times$$
$$\times C_{\frac{N+2m_-+2\delta_2-2}{4}, \frac{m_2+N_2-N_1}{2}; \frac{N-2m_-+2\delta_1-2}{4}, \frac{m_1+N_1-N_2}{2}}^{j+\frac{\delta_1+\delta_2}{2}, \frac{m_1+m_2}{2}}. \tag{31}$$

Equation (31) proves that the coefficients for the expansion of the parabolic basis in terms of the spherical basis are nothing but the analytical continuation, for real values of their arguments, of the $SU(2)$ Clebsch-Gordan coefficients.

The inverse representation has the form

$$\psi_{Njms} = \sum_{N_1, M_1 M_2} \tilde{W}_{Njms}^{N_1}(\delta_1, \delta_2)\ \psi_{N_1 N_2 M_1 M_2}. \tag{32}$$

The expansion coefficients in (32) are given by the expression

$$\tilde{W}_{Njms}^{N_1}(\delta_1, \delta_2) = (-1)^{N_1 + \frac{m-s+|m-s|}{2}}\, \delta_{M_1, m+s}\, \delta_{M_2, m-s}\ \times \tag{33}$$

$$\times\, C^{j+\frac{\delta_1+\delta_2}{2}, \frac{m_1+m_2}{2}}_{\frac{N+2m_- +2\delta_2-2}{4}, \frac{N+2m_- +2\delta_2-2}{4} - n_1; \frac{N-2m_- +2\delta_1-2}{4}, n_1+|m-s| - \frac{N-2m_- -2\delta_1-2}{4}}. \tag{34}$$

4 Prolate Spheroidal Basis

Let us determine the four-dimensional spheroidal coordinates

$$u_0 + iu_1 = \frac{d}{2}\sqrt{(\xi+1)(1+\eta)}\, e^{i\frac{\alpha+\gamma}{2}}, \qquad u_2 + iu_= \frac{d}{2}\sqrt{(\xi-1)(1-\eta)}\, e^{i\frac{\alpha-\gamma}{2}}, \tag{35}$$

where $\xi \in [1, \infty)$, $\eta \in [-1, 1]$, and d is the interfocus distance.

In the spheroidal system of coordinates the four-dimensional double singular oscillator potential has the form

$$V = \frac{\mu d^2 \omega^2}{2}(\xi + \eta) + \frac{4}{d^2}\left[\frac{c_1}{(\xi+1)(1+\eta)} + \frac{c_2}{(\xi-1)(1-\eta)}\right].$$

In the coordinates (35) Laplace operator have the form

$$\frac{\partial^2}{\partial u_i^2} = \frac{8}{d^2(\xi-\eta)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}\right] - \right.$$

$$\left. -\frac{1}{2}\left(\frac{1}{\xi+1} - \frac{1}{1+\eta}\right)\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\gamma}\right)^2 + \frac{1}{2}\left(\frac{1}{\xi-1} + \frac{1}{1-\eta}\right)\left(\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\gamma}\right)^2\right\}.$$

After the substitution

$$\psi(\xi, \eta, \alpha, \gamma) = \psi_1(\xi)\psi_2(\eta)\, e^{im\alpha}\, e^{is\gamma}$$

the variables in the Schrödinger equation (1) are separated

$$\left[\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi} + \frac{m_1^2}{2(\xi+1)} - \frac{m_2^2}{2(\xi-1)} - \frac{a^4 d^4}{16}(\xi^2-1) + \frac{\mu E d^2}{4\hbar^2}\xi\right]\psi_1 = Q\psi_1, \tag{36}$$

$$\left[\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta} - \frac{m_1^2}{2(1+\eta)} - \frac{m_2^2}{2(1-\eta)} - \frac{a^4 d^4}{16}(1-\eta^2) - \frac{\mu E d^2}{4\hbar^2}\eta\right]\psi_2 = -Q\psi_2, \tag{37}$$

where Q is a separation constant in spheroidal coordinates. By eliminating the the energy E from Eqs.(36) and (37), we produce the operator

$$\begin{aligned}
\hat{Q} \ &= \ -\frac{1}{\xi-\eta}\left\{\eta\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial}{\partial\xi}\right] + \xi\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}\right]\right\} - \\
&- \ \frac{\xi+\eta+1}{2(\xi+1)(1+\eta)}\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\gamma}\right)^2 - \frac{\xi+\eta-1}{2(\xi-1)(1-\eta)}\left(\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\gamma}\right)^2 + \\
&+ \ \frac{a^4 d^4}{16}(\xi\eta+1) + \frac{c_1(\xi-\eta)}{2(\xi+1)(1+\eta)} + \frac{c_2(\xi-\eta)}{2(\xi-1)(1-\eta)},
\end{aligned} \tag{38}$$

the eigenvalues of which are Q and the eigenfunctions of which are $\psi(\xi,\eta,\alpha,\gamma)$. The significance of the self-adjoint operator $\hat{Q}$ can be found by switching to Cartesian coordinates. Passing to the Cartesian coordinates in (38) and taking (7) and (22) into account, we obtain

$$\hat{Q} = \hat{\Lambda} + \frac{a^2 d^2}{4}\hat{\Omega}. \tag{39}$$

Therefore,

$$\hat{Q}\psi_{Nqms}(\xi,\eta,\alpha,\gamma;d,\delta_1,\delta_2) = Q_q\psi_{Nqms}(\xi,\eta,\alpha,\gamma;d,\delta_1,\delta_2), \tag{40}$$

where index q labels the eigenvalues of the operator $\hat{Q}$.

Now construct the spheroidal basis of the four-dimensional double singular oscillator using the following expressions:

$$\psi_{Nqms} = \sum_{j=m_+}^{N/2} U^{j}_{Nqms}\left(R;\delta_1,\delta_2\right)\,\psi_{Njms}. \tag{41}$$

$$\psi_{Nqms} = \sum_{N_1=0}^{(N-|M_1|-|M_2|)/2} V^{N_1}_{Nqms}\left(R;\delta_1,\delta_2\right)\,\psi_{N_1N_2M_1M_2}. \tag{42}$$

Substituting (41) and (42) into (40), and then using (39) we arrive at the following algebraic equations:

$$\begin{gathered}\left[Q_q - \left(j+\frac{\delta_1+\delta_2}{2}\right)\left(j+\frac{\delta_1+\delta_2}{2}+1\right)\right]U^{j}_{Nqms} = \frac{a^2d^2}{4}R\sum_{j'}U^{j'}_{nqms}(\hat{\Omega})_{jj'},\\[2ex] \left[Q_q - \frac{a^2d^2}{4}\left(N_1-N_2+\frac{m_1-m_2}{2}\right)\right]V^{N_1}_{Nqms} = \sum_{N_1'}V^{N_1'}_{Nqms}(\hat{\Lambda})_{N_1N_1'},\end{gathered} \tag{43}$$

where

$$(\hat{\Omega})_{jj'} = \int \psi^{*}_{Njms}\,\hat{\Omega}\,\psi_{Njms}dV, \qquad = \int \psi^{*}_{N_1N_2M_1M_2}\,\hat{\Lambda}\,\psi_{N_1'N_2'M_1M_2}\,dV.$$

Now using expansions (31), (33) and formulae [17]

$$C^{c\gamma}_{a\alpha;b\beta} = -\left[\frac{4c^2(2c+1)(2c-1)}{(c+\gamma)(c-\gamma)(b-a+c)(a-b+c)(a+b-c+1)(a+b+c+1)}\right]^{1/2}\times$$
$$\times\left\{\left[\frac{(c-\gamma-1)(c+\gamma-1)(b-a+c-1)(a-b+c-1)(a+b-c+2)(a+b+c)}{4(c-1)^2(2c-3)(2c-1)}\right]^{1/2}\times\right.$$
$$\left.\times C^{c-2,\gamma}_{a\alpha;b\beta} - \frac{(\alpha-\beta)c(c-1)-\gamma a(a+1)+\gamma b(b+1)}{2c(c-1)}C^{c-1,\gamma}_{a\alpha;b\beta}\right\},$$

$$\begin{gathered}\left[\,c(c+1)-a(a+1)-b(b+1)-2\alpha\beta\right]\,C^{c,\gamma}_{a,\alpha;b,\beta} =\\ = \sqrt{(a+\alpha)(a-\alpha+1)(b-\beta)(b+\beta+1)}\,C^{c,\gamma}_{a,\alpha-1;b,\beta+1} +\\ +\sqrt{(a-\alpha)(a+\alpha+1)(b+\beta)(b-\beta+1)}\,C^{c,\gamma}_{a,\alpha+1;b,\beta-1},\end{gathered}$$

and with the orthonormalization conditions

$$\sum_{\alpha+\beta=\gamma} C^{c\gamma}_{a\alpha;b\beta} C^{c'\gamma'}_{a\alpha;b\beta} = \delta_{c'c}\delta_{\gamma'\gamma}, \qquad \sum_{c=|\gamma|}^{a+b} C^{c\gamma}_{a\alpha;b\beta} C^{c\gamma}_{a\alpha';b\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}$$

for the Clebsch-Gordan coefficients of the group $SU(2)$, for the matrix elements $(\hat{\Omega})_{jj'}$ and $(\hat{\Lambda})_{N_1 N_1'}$ we get the expressions

$$(\hat{\Omega})_{jj'} = \frac{(m_1+m_2)(m_1-m_2)}{(2j+\delta_1+\delta_2)(2j+\delta_1+\delta_2+2)}\delta_{j',j} - \frac{2\left(A^{j+1}_{nm}\delta_{j',j+1} + A^{j}_{nm}\delta_{j',j-1}\right)}{2j+\delta_1+\delta_2}, \tag{44}$$

$$\begin{aligned}(\hat{\Lambda})_{N_1 N_1'} = \Big[&(N_1+1)(N_2+m_-) + (\frac{N}{2} - N_1 + \delta_2)(N_1 + |m-s| + \delta_2) + \\ &+ m_-(m_+ + \delta_2)\frac{1}{4}(\delta_1-\delta_2)(\delta_1-\delta_2-2) + \Big]\delta_{N_1' N_1} - \\ &- \sqrt{n_2(N_1+1)(N_1+m_1+1)(N_2+m_2)}\,\delta_{N_1',N_1+1} - \\ &- \sqrt{N_1(N_2+1)(N_1+m_1+1)(N_2+m_2+1)}\,\delta_{N_1',N_1-1}\end{aligned} \tag{45}$$

where

$$\begin{aligned} A^{j}_{Nms} &= \sqrt{(j-m_-+\delta_1)(j+m_-+\delta_2)} \times \\ &\times \left[\frac{(j-m_+)(j+m_++\delta_1+\delta_2)(N-2j)(N+2j+2\delta_1+2\delta_2)}{4\left(j+\frac{\delta_1+\delta_2}{2}\right)^2(2j+\delta_1+\delta_2-1)(2j+\delta_1+\delta_2+1)}\right]^{1/2}. \end{aligned}$$

Substituting expressions (44) and (45) into the algebraic equations (43), we derive the three-term recursion relations

$$\begin{aligned} A^{j+1}_{Nms}U^{j+1}_{Nqms} + A^{j}_{nms}U^{j-1}_{Nnqms} &= \left\{\frac{4}{a^2d^2}\left[Q - \left(j+\frac{\delta_1+\delta_2}{2}\right)\left(j+\frac{\delta_1+\delta_2}{2}+1\right)\right] - \right. \\ &- \left.\frac{(m_1+m_2)(m_1-m_2)}{(2j+\delta_1+\delta_2)(2j+\delta_1+\delta_2+2)}\right\}U^{j}_{Nqms} \end{aligned}$$

$$\begin{aligned} \Big[&(N_1+1)(N_2+m_-) + (N - N_1 + \delta_2)(N_1+m_2) + \frac{1}{4}(\delta_1-\delta_2)(\delta_1-\delta_2-2) + \\ &+ m_-(m_++\delta_2) + \frac{a^2d^2}{2}\left(N_1 - N_2 + \frac{m_1-m_2}{2}\right) - Q_q\Big]V^{N_1}_{Nqms} = \\ &= \sqrt{N_2(N_1+1)(N_1+m_1+1)(N_2+m_2)}\,V^{N_1+1}_{Nqms} + \\ &+ \sqrt{N_1(N_2+1)(N_1+m_1+1)(N_2+m_2+1)}\,V^{N_1-1}_{Nqms} \end{aligned}$$

for the interbasis expansions coefficients U^{j}_{Nqms} and $V^{N_1}_{Nqms}$.

Acknowledgements. The author is grateful to Dr. L.G. Mardoyan for many helpful dicussions.

References

[1] L.G. Mardoyan, M.G. Petrosyan, Phys. At. Nucl. **70**, 572 (2007).

[2] L.G. Mardoyan, J. Math. Phys. **44**, 4981 (2203).

[3] P. Kustaanheimo, E. Stiefel. J. Reine Angew. Math. **218**, 204 (1965).

[4] A.O. Barut, C.K.E. Schneider, R. Wilson, J. Math. Phys. **20**, 2244 (1979).

[5] M. Kibler, T. Negadi, Croat. Chem. Acta **57**, 1509 (1984).

[6] A. Nersessian, V. Ter-Antonyan, Mod. Phys. Lett. **A9**, 2431 (1994).

[7] D. Zwanziger, Phys. Rev. **176**, 1480 (1968).

[8] H. McIntosh, A. Cisneros, J. Math. Phys. **11**, 896 (1970).

[9] M. Kibler, L.G. Mardoyan, G.S. Pogosyan, Int. J. Quan. Chem. **52**, 1301 (1994).

[10] H. Hartmann, Theor. Chim. Acta **24**, 201 (1972).

[11] H. Hartmann, R. Schuch, J. Radke, Theor. Chim. Acta **42**, 1 (1976).

[12] H. Hartmann, R. Schuch, Int. J. Quant. Chem. **18**, 125 (1980).

[13] L.G. Mardoyan, Phys. At. Nucl. **68**, 1746 (2005).

[14] L.G. Mardoyan, G.S. Pogosyan, A.N. Sissakian, V.M. Ter-Antonyan: *Quantum systems with hidden symmetry* (FIZMATLIT Publ., Moscow 2006) (in Russian).

[15] S. Flügge, *Problems in quantum mechanics* (Springer-Verlag, Berlin-Heidelberg-New York 1971) Vol.1.

[16] L.D. Landau, E.M. Lifshitz, *Course of Theoretical Physics*, *Quantum Mechanics: NonRelativistic Theory* (Nauka, Moscow, 1989, 4th ed.; Pergamon Press, Oxford, 1977) vol. 3.

[17] D.A. Varshalovich, A.N. Moskalev,V.K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).